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SHORT COMMUNICATION

A CURIOUS RENEWAL PROCESS AVERAGE*

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The lifetimes of a renewal process observed during a fixed interval $(0, t]$ are smaller, on the average, than the process mean lifetime; it is shown that the mean observed lifetime has a particularly simple form.

Renewal processes
MTBF

Consider an ordinary renewal process consisting of nonnegative and independent random variables $\{X_k, k \geqslant 1\}$ with common distribution F, and density f. The random number of renewals in a fixed interval $(0, t]$, $N(t)$, has the distribution

$$P_n(t) = \mathbf{P}\{N(t) = n\} = F^{n^*}(t) - F^{(n+1)^*}(t) \quad (n \geqslant 0).$$

Where $F^{n^*}(t)$ is the n-fold convolution of F with itself. Let $M(t) = \mathbf{E}N(t)$. It is known that $\mathbf{E}\{S_{N(t)+1}\} = \mathbf{E}\{X\} \cdot [M(t)+1]$. However, simple results for $S_{N(t)}$, the sum of completed intervals in $(0, t]$, do not seem to be available; for example we have

$$\mathbf{E}\{S_{N(t)} \mid N(t) > 0\} = \frac{\int_0^t x[1 - F(t-x)]\,\mathrm{d}M(x)}{F(t)}.$$

The mean value of such an interval has the following form which is unexpectedly simple—this was discovered while investigating biases in renewal testing [1].

Theorem. *We have*

$$\mathbf{E}\left\{\frac{S_{N(t)}}{N(t)} \middle| N(t) > 0\right\} = \mathbf{E}\{X_1 \mid X_1 \leqslant t\} = \frac{\int_0^t x\,\mathrm{d}F(x)}{F(t)} = t - \int_0^t \left[\frac{F(x)}{F(t)}\right] \mathrm{d}x.$$

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Proof. We have

$$\mathbf{E}\left\{\frac{S_{N(t)}}{N(t)}\Big|N(t)>0\right\}=\frac{\sum_{n=1}^{\infty}n^{-1}\int_0^t[yf^{n^*}(y)][1-F(t-y)]\,\mathrm{d}y}{F(t)}.$$

If $\tilde{f}(s)=\int_0^{\infty}\mathrm{e}^{-st}\,\mathrm{d}F\,(t)$ is the transform of f, then the integral above is the convolution of $1-F(t)$ and $tf^{n^*}(t)$, which have transforms $[1-\tilde{f}(s)]/s$ and

$$\int_0^{\infty}\mathrm{e}^{-st}tf^{n^*}(t)\,\mathrm{d}t=-\frac{\mathrm{d}}{\mathrm{d}s}\int_0^{\infty}\mathrm{e}^{-st}f^{n^*}(t)\,\mathrm{d}t=-\frac{\mathrm{d}}{\mathrm{d}s}[\tilde{f}(s)]^n=-n\tilde{f}'(s)[\tilde{f}(s)]^{n-1},$$

respectively. Therefore, the transform of the above sum is simply

$$\frac{(-\tilde{f}'(s))[1-\tilde{f}(s)]}{s}\sum_{n=1}^{\infty}\frac{1}{n}n[\tilde{f}(s)]^{n-1}=\frac{-\tilde{f}'(s)}{s},$$

which is recognized as the transform of $\int_0^t x\,\mathrm{d}F\,(x)$. This leads to the desired result.

Remark. A simple proof which perhaps explains the theorem better was suggested by H. Gerber (University of Michigan) along the following lines: It is easily verified that, given $N(t)=n>0$, $X_1, X_1,\ldots, X_n$ are *exchangeable random variables*. Therefore, given $N(t)=n>0$, $\mathbf{E}\{S_n/n\}=\mathbf{E}\{X_1\}$; but, cases in which $n=1,2,\ldots$ are exactly those cases in which $X_1\leq t$.

The average completed interval is approximately $\frac{1}{2}t$ for small values of t, and increases monotonically to $\mathbf{E}\{X_1\}$, usually slowly. For example, if f is exponential with parameter λ, then

$$\mathbf{E}\{X_1\,|\,X_1\leq t\}=\frac{1}{\lambda}\left[\frac{1-(1+\lambda t)\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right].$$

Unfortunately, the simplicity of the above result does not generalize to other functions of $W(t)=S_{N(t)}/N(t)(N(t)>0)$. For example, the density of $W=w$ consists of different sums of convolutions over different intervals. In the exponential case, this density,

$$p(w)=\lambda\frac{\sum_{n=1}^{[t/w]}\frac{n^n(\lambda w)^{n-1}}{(n-1)!}}{(\mathrm{e}^{\lambda t}-1)},$$

is a real wonder, being level over $(\frac{1}{2}t, t]$, linear over $(\frac{1}{3}t, \frac{1}{2}t]$, quadratic over $(\frac{1}{4}t, \frac{1}{3}t]$, etc.

Acknowledgement

I would like to thank the referee for a correction.

Reference

[1] W.S. Jewell, 'Reliability growth' as an artifact of renewal testing, ORC 78-9, Operations Research Center, University of California, Berkeley (1978).